

A pathwise interpretation of the Gorin-Shkolnikov identity

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Abstract

In a recent paper by Gorin and Shkolnikov (2016), they have found, as a corollary to their result relevant to random matrix theory, that the area below a normalized Brownian excursion minus one half of the integral of the square of its total local time, is identical in law with a centered Gaussian random variable with variance $1/12$. In this note, we give a pathwise interpretation to their identity; Jeulin's identity connecting normalized Brownian excursion and its local time plays an essential role in the exposition.

1 Introduction

Let $r = \{r_t\}_{0 \leq t \leq 1}$ be a normalized Brownian excursion, that is, it is identical in law with a standard 3-dimensional Bessel bridge, which has the duration $[0, 1]$, and starts from and ends at the origin; see e.g., [1, Section (2.2)] and references therein for the definition of normalized Brownian excursion and its equivalence in law with standard 3-dimensional Bessel bridge. We denote by $l = \{l_x\}_{x \geq 0}$ the total local time process of r ; namely, by the occupation time formula, two processes r and l are related via

$$H(x) := \int_0^1 \mathbf{1}_{\{r_t \leq x\}} dt = \int_0^x l_y dy \quad \text{for all } x \geq 0, \text{ a.s.} \quad (1.1)$$

In a recent paper [2], Gorin and Shkolnikov have found the following remarkable identity in law as a corollary to one of their results:

Theorem 1.1 ([2], Corollary 2.15). *The random variable X defined by*

$$X := \int_0^1 r_t dt - \frac{1}{2} \int_0^\infty (l_x)^2 dx$$

is a centered Gaussian random variable with variance $1/12$.

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In [2], they have shown that the expected value of the trace of a random operator indexed by $T > 0$, arising from random matrix theory, admits the representation

$$\sqrt{\frac{2}{\pi T^3}} \mathbb{E} \left[\exp \left(-\frac{T^{3/2}}{2} X \right) \right]$$

for any $T > 0$; in comparison of this expression with the existing literature asserting that the expected value is equal to $\sqrt{2/(\pi T^3)} \exp(T^3/96)$ for every $T > 0$, they have obtained Theorem 1.1 by the analytic continuation and the uniqueness of characteristic functions.

In this note, we give a proof of Theorem 1.1 without relying on random matrix theory; Jeulin's identity in law ([4, p. 264], [1, Proposition 3.6]):

$$\{r_t\}_{0 \leq t \leq 1} \stackrel{(d)}{=} \left\{ \frac{1}{2} l_{H^{-1}(t)} \right\}_{0 \leq t \leq 1} \quad (1.2)$$

with

$$H^{-1}(t) := \inf \{x \geq 0; H(x) \geq t\},$$

plays a central role in the proof.

2 Proof of Theorem 1.1

In this section, we give a proof of Theorem 1.1 and provide some relevant results.

Proof of Theorem 1.1. Recall from the representation of r by means of a stochastic differential equation (see, e.g., [5, Chapter XI, Exercise (3.11)]) that the process $W = \{W_t\}_{0 \leq t \leq 1}$ defined by

$$W_t := r_t - \int_0^t \frac{ds}{r_s} + \int_0^t \frac{r_s}{1-s} ds \quad (2.1)$$

is a standard Brownian motion. We integrate both sides over $[0, 1]$ and use Fubini's theorem on the right-hand side to see that

$$\int_0^1 W_t dt = \int_0^1 r_t dt - \int_0^1 \frac{ds}{r_s} \int_s^1 dt + \int_0^1 ds \frac{r_s}{1-s} \int_s^1 dt,$$

which entails

$$\frac{1}{2} \int_0^1 W_t dt = \int_0^1 r_t dt - \frac{1}{2} \int_0^1 \frac{1-t}{r_t} dt. \quad (2.2)$$

Note that the left-hand side is a centered Gaussian random variable with variance $1/12$. By Jeulin's identity (1.2), the right-hand side of (2.2) is identical in law with

$$\frac{1}{2} \int_0^1 l_{H^{-1}(t)} dt - \int_0^1 \frac{1-t}{l_{H^{-1}(t)}} dt. \quad (2.3)$$

We change variables with $t = H(x)$, $x \geq 0$, to rewrite (2.3) as

$$\begin{aligned} & \frac{1}{2} \int_0^\infty l_x H'(x) dx - \int_0^\infty \frac{1 - H(x)}{l_x} H'(x) dx \\ &= \frac{1}{2} \int_0^\infty (l_x)^2 dx - \int_0^\infty dx \int_0^1 dt \mathbf{1}_{\{r_t > x\}} \\ &= \frac{1}{2} \int_0^\infty (l_x)^2 dx - \int_0^1 r_t dt, \end{aligned} \tag{2.4}$$

where the second line follows from the definition (1.1) of H and the third from Fubini's theorem. Combining this expression with (2.2) yields

$$\frac{1}{2} \int_0^1 W_t dt \stackrel{(d)}{=} \frac{1}{2} \int_0^\infty (l_x)^2 dx - \int_0^1 r_t dt$$

and concludes the proof. \square

We give a remark on the proof. In what follows we denote

$$M(r) = \max_{0 \leq t \leq 1} r_t.$$

Remark 2.1. (1) We see from the above proof that the random variables

$$\int_0^1 r_t dt, \quad \frac{1}{2} \int_0^\infty (l_x)^2 dx, \quad \frac{1}{2} \int_0^1 \frac{1-t}{r_t} dt$$

have the same law; they are also identical in law with

$$\frac{1}{2} \int_0^t \frac{t}{r_t} dt$$

by the time-reversal: $\{r_{1-t}\}_{0 \leq t \leq 1} \stackrel{(d)}{=} \{r_t\}_{0 \leq t \leq 1}$. The Laplace transform of the law of $\int_0^1 r_t dt$ is given in [3, Lemma 4.2] and [1, Proposition (5.5)] in terms of a series expansion.

(2) We see from (1.1) that

$$\int_0^\infty l_y dy = \int_0^{M(r)} l_y dy = 1.$$

Therefore, to be more specific, the second integral in (2.4) should be written as

$$\int_0^{M(r)} \frac{1 - H(x)}{l_x} H'(x) dx.$$

Using the same reasoning as the above proof, we may obtain the following extension of Theorem 1.1:

Proposition 2.1. *For every positive integer n , the random variable*

$$2 \int_{[0,1]^n} \min \{r_{t_1}, \dots, r_{t_n}\} dt_1 \cdots dt_n - \frac{n+1}{2} \int_0^\infty (1-H(x))^{n-1} (l_x)^2 dx$$

has the Gaussian distribution with mean zero and variance $1/(2n+1)$.

Proof. For each fixed n , we multiply both sides of (2.1) by $(1-t)^{n-1}$ and integrate them over $[0, 1]$. Then using Fubini's theorem, we obtain

$$\int_0^1 (1-t)^{n-1} W_t dt = \frac{n+1}{n} \int_0^1 (1-t)^{n-1} r_t dt - \frac{1}{n} \int_0^1 \frac{(1-t)^n}{r_t} dt. \quad (2.5)$$

Since the left-hand side may be expressed as $(1/n) \int_0^1 (1-t)^n dW_t$, we see that it is a centered Gaussian random variable with variance

$$\frac{1}{n^2} \int_0^1 (1-t)^{2n} dt = \frac{1}{n^2(2n+1)}.$$

On the other hand, by Jeulin's identity (1.2), the right-hand side of (2.5) is identical in law with

$$\begin{aligned} & \frac{n+1}{2n} \int_0^\infty (1-t)^{n-1} l_{H^{-1}(t)} dt - \frac{2}{n} \int_0^1 \frac{(1-t)^n}{l_{H^{-1}(t)}} dt \\ &= \frac{n+1}{2n} \int_0^\infty (1-H(x))^{n-1} (l_x)^2 dx - \frac{2}{n} \int_0^{M(r)} (1-H(x))^n dx. \end{aligned}$$

By (1.1), we may rewrite the integral in the last term as

$$\begin{aligned} \int_0^{M(r)} dx \left(\int_0^1 dt \mathbf{1}_{\{r_t > x\}} \right)^n &= \int_0^{M(r)} dx \int_{[0,1]^n} dt_1 \cdots dt_n \prod_{i=1}^n \mathbf{1}_{\{r_{t_i} > x\}} \\ &= \int_{[0,1]^n} \min \{r_{t_1}, \dots, r_{t_n}\} dt_1 \cdots dt_n, \end{aligned}$$

where we used Fubini's theorem for the second equality. Combining these leads to the conclusion. \square

We end this note with a comment on a relevant fact deduced from the proof of Proposition 2.1.

Remark 2.2. It is well known (see, e.g., [1, Equation (5d)]) that

$$M(r) \stackrel{(d)}{=} \frac{1}{2} \int_0^1 \frac{dt}{r_t};$$

in fact, Jeulin's identity (1.2) entails that

$$\frac{1}{2} \int_0^1 \frac{dt}{r_t} \stackrel{(d)}{=} \int_0^{M(r)} \frac{1}{l_x} \times l_x dx = M(r).$$

Combining this fact with a part of the proof of Proposition 2.1, one sees that the sequence of random variables

$$M(r), \quad \int_{[0,1]} r_t dt, \quad \int_{[0,1]^2} \min \{r_{t_1}, r_{t_2}\} dt_1 dt_2, \dots$$

is identical in law with

$$\frac{1}{2} \int_0^1 \frac{(1-t)^n}{r_t} dt, \quad n = 0, 1, 2, \dots,$$

as well as with

$$\frac{1}{2} \int_0^1 \frac{t^n}{r_t} dt, \quad n = 0, 1, 2, \dots$$

by the time-reversal.

References

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